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HEAT CONDUCTION AND HEAT EXCHANGE IN TECHNOLOGICAL PROCESSES

STRUCTURAL PROPERTIES OF DYNAMIC SYSTEMS AND INVERSE PROBLEMS OF MATHEMATICAL PHYSICS

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A direct relationship between the theory of inverse problems of mathematical physics and the theory of structural properties of dynamic systems is established based on which the inverse problems of mathematical physics and heat conduction are classified and some of the works on them are reviewed.

Introduction. The foundations of the theory of inverse problems of mathematical physics were laid in the 1950s–1960s in the works of A. N. Tikhonov, M. M. Lavrent'ev, I. M. Gel'fand, B. M. Levitan, M. G. Krein, V. A. Marchenko, L. D. Faddeev, and many other mathematicians. The special properties of inverse problems are that, unlike primal problems, they do not possess the property of correctness in the sense of Adamar. In this connection, A. N. Tikhonov and his followers have developed the theory of regularization of ill-posed problems and have proposed stable methods of their solution [1–12].

In thermophysics, inverse problems occur as problems of either diagnostics of the thermophysical parameters and internal and (or) boundary sources of the processes of transfer or control and synthesis of the above parameters and sources. We emphasize that the "investigation methodology based on solution of inverse problems is one new line in studying heat- and mass-exchange processes and in processing and optimizing thermal regimes of technical objects and technological processes [13]." The problems and methods of solution of the inverse problems of heat exchange have been presented in detail in [14] (this monograph is now classical).

In the present work, we review the basic classes of inverse problems of mathematical physics and, in particular, inverse problems of heat conduction; the inverse problems are organized in accordance with the classification (proposed in [15]) of inverse problems of mathematical physics. The basis for the classification used is the scheme of cause-and-effect relations of dynamic systems. The notion of a dynamic system is fundamental for primal problems of mathematical physics. The theory of structural properties and characteristics of systems, such as controllability, observability, reversibility, realizability, and others, has also been developed within the framework of dynamic systems [16–29]. It turned out that these characteristics are directly related to the formulation of a number of classical inverse problems of mathematical physics and inverse problems of heat conduction [15]. Thus, the classification of inverse problems of mathematical physics that is presented in this review links the theory of inverse problems to the theory of dynamic systems in the space of states, which contributes to the interdisciplinary exchange of results and, in particular, to the use of the methods of the theory of dynamic systems in the theory of inverse heat-conduction problems.

It should be noted that investigations of the inverse problems of mathematical physics are the focus of numerous works, including monographs (see, e.g., [1, 2, 8, 13, 14, 30–56]). Therefore, in this review, we have restricted our consideration to only the part of the works that were not mentioned in the above monographs but are of interest from the viewpoint of the theory of systems and inverse problems of mathematical physics. In this work, we do not consider inverse problems in the stationary formulation and a wide class of problems associated with the shaping of bodies.

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Basic Notions of System Theory. In describing inverse problems in general form, it is expedient to use the set-theoretical apparatus of a mathematical abstract system theory [16–18, 57]. The result of the abstract system theory, best suited to our purposes, is rigorous formalization of the notion of cause-and-effect relations of systems considered within the framework of some mathematical models or others. Therefore, the basis for the classification of inverse problems used here is the scheme of cause-and-effect relations of an abstract dynamic system. Since most of the inverse problems of mathematical physics are currently described in terms of distributed dynamic systems [2, 14, 30–35], the classification developed further may serve as a basis for the hierarchic structure with a more detailed taxonomy of inverse problems and inverse heat-conduction problems, for example, in thermophysical and other signs [31–33, 35, 41, 48, 58–60].

We briefly describe set-theoretical structures of the abstract system theory that will be necessary in the future. This will enable us to emphasize the fundamental character and universality of such notions as "dynamic system," "input," "state," "output," and "reaction of the system (input–output map)." In turn, the basic system properties — controllability, observability, reversibility, realizability, and structural and parametric identifiability — are adequately described precisely in terms of the above notions and are of crucial importance in formulating and classifying the inverse problems of mathematical physics. The algebraic methods of the abstract system theory, thus far developed only for linear systems in detail, are also of great interest [16, 19–22, 61]. We note that assimilation of algebraic system-structural methods by the heat-transfer theory has been reflected in [19, 62].

In the general case [17], the abstract system S is determined as a subset of the Cartesian product $\Omega \times \Gamma$ of certain sets Ω and Γ:

$$S \subseteq \Omega \times \Gamma \,. \tag{1}$$

The components Ω and Γ of the Cartesian product $\Omega \times \Gamma$ are respectively called the input and output objects of the system S. For time-variable systems the input and output objects are sets of functions of the time t, $t \in \Theta$, i.e., $\Omega = U^{\Theta}$ and $\Gamma = Y^{\Theta}$, where U is the set of values of the input quantities, Y is the set of values of the output quantities, and Θ is the linearly ordered set of the instants of time. The set Θ can be discrete or continuous; also, we do not rule out a combined variant. The functions of Ω are called the inputs of the system S, whereas the functions of Γ are the outputs.

Relation (1) may be interpreted as a generally multivalued map S: $\Omega \to \Gamma$ setting up a correspondence between the causes (inputs) and effects (outputs). The multivaluedness of S is related to certain internal parameters of the system. In the abstract system theory, these parameters are called the states of a system [16, 17]. For a prescribed system S we can always construct the object of global states — such a set X that the state $x \in X$ ensuring the equality $\gamma = R(x, \omega)$ exists for any input–output pair (ω, γ), $(\omega, \gamma) \in S$. The map R: $X \times \Omega \to \Gamma$ is the global reaction of the system S [17].

The notion of a time-causal system holds one central position in the abstract system theory. Next, following [16], we identify the notion of a causal system and a dynamic system. The future behavior of a dynamic system exerts no influence on its past, and this imposes certain restrictions on the structure of the map of R. Such restrictions are usually described [16, 17] in terms of the transient function, the output map, and the contraction $R_{[\tau,t]}$ of the global reaction to the time interval $[\tau, t]$, where τ is the initial instant of time and t is the running instant.

Let us denote the contraction of the input $u(\cdot)$ to the time interval $[\tau, t]$ by $u_{[\tau,t]}$. Then the value of the transient function $\varphi : \Theta \times \Theta \times X \times \Omega_{[\tau,t]} \to X$ coincides with the state

$$x(t) = \varphi(t; \tau; x(\tau), u_{[\tau,t]}) \tag{2}$$

of the dynamic system at the instant of time t if the dynamic system was in the state $x(\tau)$ at the initial instant τ, $\tau < t$, and the input $u_{[\tau,t]}$ acted on it. The transient function possesses a number of characteristic properties [16] that are not the focus of our attention here.

The output map η: $\Theta \times X \times U \to Y$ determines the value of the output at the running instant of time t:

$$y(t) = \eta(t, x(t), u(t)) \,. \tag{3}$$

The basic property of the reaction R of a dynamic system is that, at a prescribed instant of time, we can represent $R_{[\tau,t]}$ as the superposition of the output map (3) and the transient function (2):

$$y(t) = R_{[\tau,t]}(x, u) = \eta(t, \varphi(t; \tau; x(\tau), u_{[\tau,t]}), u(t)) . \quad (4)$$

Mathematical models of specific systems are prescribed, as a rule, by a generating system of equations that may be considered as an algorithm enabling one to obtain the transient function of the object under study. The generating system possesses the property of time localizability. It links the nearest future of a system to the running state and the running input. For example, for discrete dynamic systems, when $t = k$, $k \in \Theta : \{..., -1, 0, 1, ...\}$, the generating system has the form

$$x(k+1) = A(k, x(k), u(k)) , \quad y(k) = \eta(k, x(k), u(k)) ,$$

for continuous dynamic systems, it has the form

$$\frac{\partial x}{\partial t} = A(t, x(t), u(t)) , \quad y(t) = \eta(t, x(t), u(t)) .$$

In what follows, the map $A : \Theta \times X \times U \to X$ will be called generating.

In the theory of energy, momentum, and material transfer, the role of states is played by the internal parameters of processes (they are often called the parameters of relaxation of processes) [63]: temperature and electromagnetic fields, the concentration distributions of different substances, the coordinates of chemical reactions, the parameters of order, and microstructural parameters, for example, the energies of rotational and vibrational degrees of freedom of atoms and molecules. In description of transfer processes using stochastic models, the system notion of state corresponds to single- or multiparticle distribution functions of microstates.

The inputs in physically realizable dynamic systems are determined by volume and boundary sources and (or) by energy, momentum, and material sinks. As far as the outputs are concerned, in considering diagnostics inverse problems, they are informative signals arriving from the sensors of recording of the transfer processes. A specific form of the output map η essentially depends on the method of measurement and on the scheme of primary processing and commutation of measurement results. For example, for the case of measurement of the temperature $T(r, t)$ at the point r_0 we have

$$y(t) = \eta(T) \equiv T(r_0, t) .$$

In the more general case of measurement of integral characteristics [25, 64–69], the form of the output map is postulated by the equation

$$y(t, r) = \int_{\omega_0} p(r, r', t)\, T(r', t)\, dr' ,$$

where the map $p(r, r', t)$ should be interpreted as the instrument function of the measuring system; here $\omega_0 \subseteq \omega \subseteq R^3$, ω being the domain of specification of the temperature field.

The independent linear dynamic systems with lumped parameters

$$S_{\text{lmpd}} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) , \\ y(t) = Cx(t) + Du(t) , \end{cases}$$

where $x \in R^n$, $u \in U = R^r$, $y \in Y = R^m$, $A : X \to X$, $B : U \to X$, and $C : X \to Y$ are the linear operators (n, r, and m are the dimensions of spaces), have been studied most adequately at present. The use of the methods of the theory of lumped dynamic systems assumes that the initial models with distributed parameters are replaced by a simplified lumped model at the early stages of calculation. Such an approach is characterized by certain drawbacks, for example, the conditions of observability, controllability, and reversibility are determined by the approximation method now and

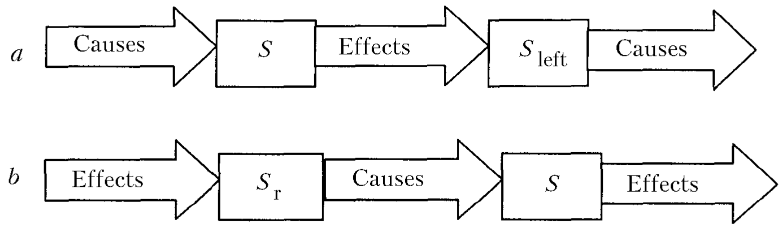

Fig. 1. Problems of diagnostics $S_{\text{left}}\ Su = u$ (a) and synthesis $SS_{\text{r}}\ y = y$ (b).

not only by the structure of the initial system. The alternative approach implies that approximation of the problem is carried out at the final stage of solution after the qualitative investigation and possible transformation of a distributed mathematical model. This approach assumes that the structural properties of distributed dynamic systems are thoroughly studied.

A mathematical model of an independent linear distributed dynamic system can be described by the system of equations

$$S_{\text{d}}: \begin{cases} \dfrac{\partial x}{\partial t} = Ax\,(t) + Bu\,(t)\,, \\ y\,(t) = Cx\,(t) + Du\,(t)\,. \end{cases}$$

Unlike the S_{lmpd} model, the space of states X of the dynamic system S_{d} is infinite dimensional and, consequently, the operator A acts in an infinite-dimensional space now. In [22, 61], one can find conditions for the X, U, and Y spaces and the operators A, B, and C that ensure a continuous dependence of the transient function

$$\varphi\,(t;\,\tau;\,x\,(\tau),\,u_{[\tau,t]}) = \exp\,(A\,(t-\tau))\,x\,(\tau) + \int_{\tau}^{t} \exp\,(A\,(t-s))\,Bu\,(s)\,ds$$

and the reaction $R(x,\ u) = Cx + Du$ on the inputs $u(\cdot)$ and the initial conditions $x(\tau)$. The one-parametric family of operators $\exp\,(At)$ forms in this case the so-called semigroup of the class C_0 [61]. We note that for linear dynamic systems describing the processes of heat transfer, the corresponding semigroup is usually expressed by the Green function $G(r,\ r',\ t)$:

$$\exp\,(A\,(t-\tau))\,T\,(\tau) = \int_{\omega} G\,(r,\,r',\,t-\tau)\,T\,(r',\,\tau)\,dr'\,.$$

Structural Properties of Dynamic Processes and Classification of Inverse Problems of Mathematical Physics. In system theory, the structural properties of dynamic systems are determined using the notions of controllability, observability, reversibility, realizability, and others [16, 19–24, 27, 29, 61, 70, 72]. These notions can be used for formulation and classification of inverse problems associated with the inversion of cause-and-effect chains of a dynamic system.

First of all, we note that the inversion of cause-and-effect relations allows dual interpretation. First, using the procedure of inversion one can determine the causes from the effects known, for example, from experiment. The second interpretation involves the possibility of synthesizing causes that ensure the required effects. In what follows, the above two inverse problems will be called concomitant ones. In system theory, the first of the concomitant problems is commonly called the problem of observation (diagnostics, reconstruction, identification), whereas the second one is called the problem of control (synthesis, design).

The problems of diagnostics and synthesis have different physical meanings, and their solution assumes different instrument realizations. Mathematically, the concomitant inverse problems are identical only in the case where the left-hand and right-hand inverse operators exist and coincide in the ratio S describing the relationship between the causes and the effects (Fig. 1). A determining property enabling us to solve, in principle, the problem of diagnostics

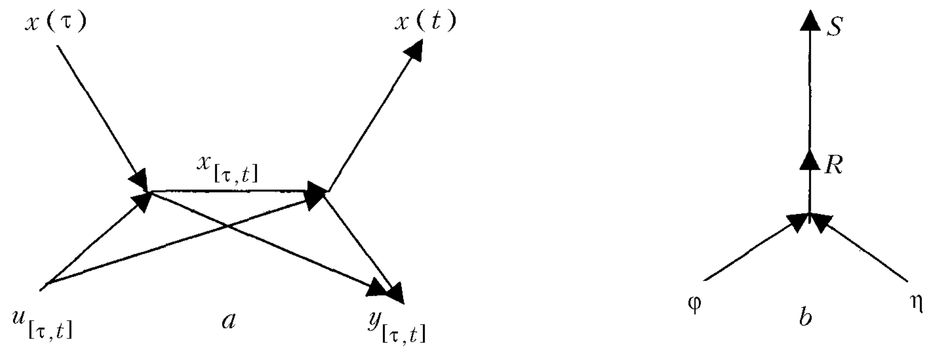

Fig. 2. Graphs of traversal of the dynamic system by the signals (a) and of internal structure of the dynamic system (b).

is the uniqueness of the solution, whereas the existence of the solution of inverse problems is sufficient for solvability of the problems of control. The key terms "observability" and "reconstructability" are used in the abstract system theory to denote the uniqueness of the solution of inverse problems. The dual term "controllability" means the existence of solutions for the problem of control.

To classify concomitant inverse problems we consider the graph of traversal of a dynamic system by the signals (Fig. 2a) and the graph reflecting the internal structure of the dynamic system (Fig. 2b), which follow from relations (2)–(4).

Inversion of the chain $x(\tau) \to x(t)$ leads to the formation of the following concomitant inverse problems. The first one is the diagnostic problem of final observation of states, i.e., the problem of reconstruction of the initial state $x(\tau)$ from the results of measurement of the final state $x(t)$. This inverse problem is sometimes called retrospective. The second of the concomitant inverse problems represents a problem of synthesis of the initial (starting) state $x(\tau)$, ensuring the required final state $x(t)$.

In an analogous manner, we can consider inversion of the cause-and-effect relations $x(\tau) \to y_{[\tau,t]}$, $u_{[\tau,t]} \to y_{[\tau,t]}$, $u_{[\tau,t]} \to x(t)$, and $(\varphi, \eta) \to R \to S$. The corresponding concomitant inverse problems are briefly described in Table 1. The inverse problems presented here hold a central position in mathematical physics in theoretical significance and volume of the existing and possible applications. The class of inverse problems can be extended further by combining the a priori information on dynamic systems and reconstructed or controlled objects. For example, information on the input (strong observability [23]) may be lacking in the problem of observation, and information on the initial state (reversibility with an unknown initial state [25]) may be lacking in the problem of reconstruction of inputs. Variants of combination of the basic inverse problems that are of applied interest are also possible [8, 18, 30, 44].

In what follows, we briefly characterize the inverse problems given in Table 1.

Retrospective Problem. This problem is associated with time inversion and can be solved using the replacement $t = -s$. The difficulty of substantive interpretation of such a replacement, which is due to the incorrectness of inverse problems, is significant for dynamic systems describing the processes of material and energy diffusion [72]. Historically, the retrospective problem has played an important role as a model of an ill-posed problem of mathematical physics [73]. The class of such problems includes, for example, the problems of restoration of the initial states of nucleation in the thermodiffusion chambers [74]. The problem on modeling of thermal convection when the process is described by a system of evolution equations and the algorithm of numerical calculations is oriented toward the use of concurrent computers is considered in inverse time in [75]. Numerical methods of solution of retrospective problems for different classes of distributed systems are presented in [12, 44, 76, 77].

Problem of Observation. The problem of observation of states of dynamic systems from the results of measurements of accessible outputs was considered for the first time in control theory [16, 19, 20, 23, 61] in connection with the construction of closed asymptotically stable regulating systems. However, the meaning of the notion of observability goes beyond the position that it holds in control theory. It is common knowledge [21, 33, 34, 78–80] that some methods of solution of the problem of observation of states possess certain universality and invariance to formulation of inverse problems.

In investigating the properties of observability of the independent linear systems S_{lmpd} and S_{d}, we can fix the input without loss of generality, in particular, we can assume that $u(t) \equiv 0$; then the output of the dynamic system S_{d} takes the form

$$y(t) = C \exp(At)\, x(0) \quad (\tau = 0)\,. \tag{5}$$

The dynamic system S_{d} is called that observed on the interval $[0, t_0]$, $t_0 > 0$, if the condition $y(t) = 0$ $\forall t \in [0, t_0]$ yields the condition $x(t) = 0$ $\forall t \in [0, t_0]$. Determination of the observability ensures a single-valued solvability of Eq. (5) for the initial state $x(0)$. However, the actual restoration of the state $x(0)$ may be made difficult by the incorrect character of solvability of (5). In particular, to reconstruct the initial-temperature distribution from the results of measurement of the temperature at the point r_0 in the linear case we must solve the integral equation of the first kind

$$y(t) = \int_{\omega} G(r_0, r', t)\, T_0(r')\, dr'\,. \tag{6}$$

Equation (6) is not correct in the sense of Hadamard even in the case of a single-valued solvability [1]. The situation is somewhat more favorable when scanning temperature sensors can be used; then we have $y(t) = T(\beta(t), t)$. For a wide class of analytical scanning functions $\beta(t)$, the problem of observation of the initial state is solvable and conditionally correct in the sense of A. N. Tikhonov. This result has been proved in [2] by the method of analytical continuation of functions of a complex variable for parabolic equations with constant coefficients.

Group-theoretical methods of investigation of the observability of nonlinear lumped dynamic systems have been reviewed in [78]. Examples of investigation of the property of observability of lumped models of thermal systems are given in [34, 80]. Variational methods of solution of the problems of observability have been considered in [12, 56]. The property of observability enables one to evaluate the running state of a dynamic system in real time using the so-called dynamic observers (identifiers of states) [16, 20, 33, 34, 79]. The method of dynamic observers is universal in a sense for solution of inverse problems, since the unknown parameters of the dynamic system can be included in the extended space of states [34, 79]. Numerical experiments published by some authors (e.g., [42, 80]) confirm the efficiency of the methods of dynamic observers for solution of inverse heat-conduction problems.

Problem of Inversion. The problem of left inversion of dynamic systems is associated with the reconstruction of the unknown inputs of a system from the results of measurement of the functionals determined on the system's running states [14, 23–25, 27–29, 31–36, 44, 47, 67–70, 80–92]. The signals sought can be both internal actions and those external relative to the object of action under study: time-varying amplitudes of the heat sources and sinks [25, 41–45, 84, 85, 87–89, 92–98], boundary temperatures [14, 31–35, 99], boundary heat loads (fluxes) [14, 31–35, 67, 82, 83], and time-variable contact resistances [31, 42]; instrument inverse problems [1, 31, 100] whose subject-matter is reconstruction of the true signal from the readings of the device also belong to the class of diagnostic problems of reconstruction of the inputs of dynamic systems. For example, a method of reconstruction of the intensity of heat release resulting from the friction of cylindrical mates is proposed in [101]. Mathematically, the problem lies in determining the term in a nontraditional boundary condition of mating for a two-dimensional heat-conduction equation with cylindrical symmetry. The coefficients of the equation, the initial and boundary conditions, and the known value of temperature on a certain curve inside the modeling domain are a priori information for this problem. Solution is based on minimization of the standard residual.

Typical properties inherent in the problem of inversion of linear systems are most easily revealed using the lumped dynamic systems S_{lmpd} as an example. Reversibility conditions for the dynamic systems S_{lmpd}, enabling us to judge the uniqueness of the solution in the case of a diagnostic problem or the existence of the solution in the problem of synthesis of control, have been indicated in [23–25, 28, 34, 69, 70]. The methods of inversion of linear lumped dynamic systems have been developed in [23–25]. A characteristic feature of these methods is representation of the inverse system in the space of states. Such representations are useful for qualitative and numerical investigation of inverse problems [25, 67, 68, 81, 92, 102]. In the case of linear distributed systems the inverse systems belong to the class of integro-differential equations with nonclassical boundary conditions [67–69, 81, 88, 92, 102].

In [67, 81], the method of inversion of linear dynamic systems has been used for solution of the problem of reconstruction of boundary heat fluxes from the results of measurement of the temperature by a differential thermocouple. Analytical and asymptotic solutions of inverse problems have been obtained; the minimum (sufficient for reconstruction of the fluxes) volume of information on initial conditions and internal sources has been indicated.

The problem of inversion of nonlinear dynamic systems have been considered in [78]. Reversibility criteria and methods of inversion of linear distributed systems have been obtained in [70]. The reversibility of distributed dynamic systems with an unknown initial state has been studied in [25] in the context of the problem of synthesis of integral characteristics that are nearly invariant to the actions induced by the system's initial state. Also, the well-known technique of conversion of boundary conditions [82], which enables us to determine the heat flux at one boundary from the temperatures and fluxes measured on the remaining part of the boundary, is easily interpreted within the framework of the method of inversion of dynamic systems. The corresponding inverse dynamic system is obtained from the direct dynamic system by simple re-marking of inputs and outputs. In particular, the problem of reconstruction of the heat-flux density on one side of the wall from the measurements of the temperature and the heat-flux density on the opposite side represents a typical example of a bilateral distributed dynamic system with an unknown initial state. The exact solution of the inverse problem has the form of a weighted infinite sum of the derivatives (increasing in order) of the observed outputs of the initial dynamic system. Substantiation of such a representation of the solution of the inverse problem in the case of analytical input functions follows from the well-known theorem of S. Kovalevskaya and, in the general case, from the theory of pseudodifferential operators with an analytical symbol [103].

The use of the term "exact analytical solution" is somewhat conditional for ill-posed inverse problems, since such a solution may be used in practice only after the corresponding regularization [1]. In the "naive approach" to the problem of regulation, one usually carries out presmoothing of observation data and restriction of the sum representing the solution to some of the first terms.

Numerical algorithms and corresponding programs meant for solution of one-dimensional inverse problems of reconstruction of boundary heat loads in linear and nonlinear media have been described in [36]. The method of parametric identification of heat fluxes that involves combined use of the method of dynamic observers (recurrent filtration) and spline approximation of the quantities sought has been considered in [33, 34, 41, 42, 80]. A review and cases of use of methods of iterational regularization [6, 13, 14, 32] for evaluation of internal sources in linear heat-conduction problems can be found in [104].

The concomitant problem of control of the outputs of a dynamic system [27–29, 55, 86, 87, 89, 91] occurs in controlling the process of drying and defrosting, artificial hyperthermia, electron-beam, plasma, laser, and induction welding and quenching of metals, transfer of heat in fuel element, zone furnaces, etc. The field of application of this inverse problem is constantly extending.

Problems of Final Reconstruction of Inputs and Control of State. In our systematics of inverse problems of mathematical physics, problems of this kind include those of reconstruction of the time functions of sources (sinks) on the right-hand sides of equations and of boundary conditions from prescribed values of the solution of the primal problem at the final instant of time. The uniqueness of the solution of inverse problems for a parabolic equation that lies in reconstructing the function of the source from the initial and boundary conditions and the solution known at the final instant of time is set up in [105]. A characteristic problem of final reconstruction of inputs is the well-known inverse problem on "historical climate" [31], associated with the study of the history of formation of the permafrost zone on the earth's surface. The problem of reconstruction of the surface temperature of a glacier from the data of measurements of the temperature in a well is considered in [106]. Mathematically, it represents an inverse problem lying in determining the boundary condition from the solution known at the final instant of time. The thermal conductivity and thermal diffusivity, the geothermal flow at the glacier's base (one boundary condition), the rate of sedimentation, and the vertical velocity of annual layers in the glacier (convective term in the heat-conduction equation) are a priori data for this problem. Determination of the boundary condition is based on Tikhonov's procedure of regularization.

The diagnostic problem of final reconstruction of inputs is accompanied by the classical problem of control of the states of a dynamic system. Different aspects of this problem have been considered in [16–22, 107, 108].

Realization of a Dynamic System. The problem of realization of dynamic systems represents an abstract formulation of the scientific approach to construction of mathematical models [26] and is a cornerstone of the abstract system theory [16, 17, 22, 61, 71]. In the formulation following from Table 1, the problem of realization lies in de-

TABLE 1. Classification of Inverse Problems of Mathematical Physics

Cause-and-effect relation	Concomitant diagnostic problem	Concomitant problem of synthesis	Reconstructed object of the diagnostic problem or controlling object of the problem of synthesis	Observed object of the diagnostic problem or object of control of the problem of synthesis	A priori information
$x(\tau) \to x(t)$	Final observation of states (retrospective problem)	Starting control of states	Initial state $x(\tau)$	Final state $x(t)$	Input $u_{[\tau,t]}$, transient function φ
$x(\tau) \to y_{[\tau,t]}$	Observation of states	Starting control of output	Initial state $x(\tau)$	Output $y_{[\tau,t]}$	Input $u_{[\tau,t]}$, reaction $R_{[\tau,t]}$
$u_{[\tau,t]} \to y_{[\tau,t]}$	Reconstruction of inputs (left inversion of the dynamic system)	Control of outputs (right inversion of the dynamic system)	Input $u_{[\tau,t]}$	Output $y_{[\tau,t]}$	Initial state $x(\tau)$, reaction $R_{[\tau,t]}$
$u_{[\tau,t]} \to x(t)$	Final reconstruction of inputs	Control of states	Input $u_{[\tau,t]}$	Final states $x(t)$	Initial state $x(\tau)$, transient function φ
$(\varphi, \eta) \to R_{[\tau,t]}$, $R_{[\tau,t]} \to S$	Realization of the dynamic system	Reaction of the dynamic system (transient function and output map)	Reaction $R_{[\tau,t]}$ or transient function and output map	Ratio S	

termining the generating map A and the output map η and it is known [16] to have a whole family of solutions. The existence of more than one dynamic system realizing the ratio S is associated with the arbitrariness in selection of the object of states. Therefore, of practical interest is the so-called problem of minimum realization, when the least, in a sense, object of states of the dynamic system is used.

The theory of minimum realization has been considerably developed for linear lumped dynamic systems [16] and for certain classes of independent linear infinite-dimensional dynamic systems [22, 61, 71]. For linear systems the minimum realization of the reaction $R_{[\tau,t]}$ is unique accurate to isomorphic transformations of the space of states. The space of states and the structure of physical systems are usually prescribed in advance; then construction of the minimum realization of a dynamic system may be considered as solution of the problem of parametric identification [21, 79] lying in reconstructing the generally unknown functional parameters of the generating and (or) output maps of the dynamic system.

In developing this range of problems, the so-called inverse spectral problem of Sturm and Liouville was of considerable importance [3, 30, 49, 50, 54]. The results of the theory of inverse spectral problems are used at present in solving the inverse problems of quantum mechanics and inverse heat-conduction problems, in seismic prospecting, in problems of synthesis of wave guides, and in others [31, 46, 51–53]. The relationship between the theory of realization of linear dynamic systems and the Sturm–Liouville inverse spectral problem has been considered in [71]. The method of boundary control of investigation of the inverse problems of mathematical physics and problems of realization of dynamic systems has been developed in [109].

The most important class of problems of realization of dynamic systems is associated with reconstruction of relaxation kernels in Volterra-type integro-differential equations [110, 111]. These inverse problems largely determine the methods of diagnostics and synthesis of viscoelastic properties of materials and, in a broader context, of media with a memory [112–116].

In the literature, one usually links the class of problems of parametric identification of dynamic systems to determination of thermophysical characteristics [13, 14, 32–35, 41–43]. In problems of material transfer, inverse problems of this kind are associated with reconstruction of the coefficients of the first or second derivatives. Papers [117–119] are devoted to the problem of existence and uniqueness of the coefficients of inverse problems. The methods of identification of the thermal conductivities (diffusivities) dependent on time and space coordinates or constants are studied for a parabolic equation in [120–126]. A characteristic problem of this kind is solved in [124], where Avdeev et al. identify the piezoconductivity of an oil bed in the model of a hydrodynamic method of probing of oil-bearing areas which, for the parabolic equations considered, is the coefficient of the second derivative of pressure. The problem is investigated in a one-dimensional formulation and it is assumed that the piezoconductivity is piecewise continuous. Classical conjugation conditions are specified at the point of discontinuity of the coefficient. The initial data for solution of the problem are the curve (known from the field experiment) of variation of bottom-hole pressure and the pressure at the external boundary. The algorithm of solution is based on minimization of target functionals and has been tested using model data. The turbulent diffusivities of the atmospheric boundary layer in a mathematical model associated with investigation of the processes of propagation of contaminants in air are identified for multidimensional problems in [126]. The process of propagation of the concentrations of the impurities is described by a three-dimensional system of diffusion equations and is anisotropic in character — the diffusivities are constant in two directions and are equal, whereas the value of the vertical turbulent diffusivity may substantially depend on altitude. Reconstruction of the turbulent diffusivity is a result of the minimization of the residual functional by the method of stochastic approximation.

The subject matter of identification of nonlinear coefficients in the equations of mathematical physics is being intensely developed [13, 14, 33–35, 41, 44, 127–131]. For example, the inverse problem on determination of the thermal conductivity dependent on the independent variables t, x, and y and on the solution is solved in [127]. A two-dimensional equation with homogeneous boundary conditions is considered. It is assumed that the process is isotropic in character. The input data are the values of the solution and the coefficient at different instants of time at different points of the modeling domain. Determination of the coefficient is reduced to finding the approximate minimum of the corresponding functional. The problem of modeling of flow of a homogeneous fluid in an inhomogeneous bed is solved numerically by the method proposed in [127]. The input data are determined on wells. The results of reconstruction of the coefficient are given.

Substantial progress has been made in solution of inverse coefficient problems for the Stefan problem [39, 132]. The inverse coefficient problem associated with a one-dimensional single-phase Stefan problem and lying in determining the thermal diffusivity, the temperature distribution, and the position of the phase front based on the classical formulation of the Stefan problem and the additional condition at the known boundary is considered, for example, in [132].

Conclusions. In mathematical system theory, much attention is given to the structural properties of dynamic systems: controllability, observability, reversibility, realizability, and others. These properties have a clear physical meaning and are often used in analyzing and synthesizing automatic regulating systems. An important mathematical feature of the structural properties of dynamic systems is their invariance relative to nongenerate transformations in the space of states and transformations of the type of feedbacks. On the other hand, the theory of inverse problems of mathematical physics, widely used in different divisions of physics and engineering, continues to intensely develop.

In the present work, we have attempted to consider in combination the structural properties of dynamic systems and inverse problems of mathematical physics with the aim of unifying a number of formulations of the inverse problems and their new classification based on consideration of the cause-and-effect relations of the dynamic systems.

NOTATION

A, generating map; $G(r, r', t)$, Green function; $R_{[\tau,t]}$, reaction of the system; S, abstract system; $S_{\rm d}$, independent linear distributed dynamic system; $S_{\rm lmpd}$, independent linear dynamic system with lumped parameters; t, running instant of time; $T(r, t)$, temperature; U, set of values of the input quantities; $u_{[\tau,t]}$, input; $x(\tau)$, initial state of the object; $x(t)$, final state of the object; Y, set of values of the output quantities; $y_{[\tau,t]}$, output; $\beta(t)$, class of analytical scanning functions; η, output map; Θ, linearly ordered set of the instants of time; τ, initial instant of time; φ, transient function; ω, domain of specification of the temperature field; $\Omega \times \Gamma$, Cartesian product of the sets of Ω and Γ. Subscripts: left, left; r, right; d, distributed; lmpd, lumped.

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